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Why Piecewise Linear Functions Are Dense in $C[0, 1]$

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In this article, we give a Stone–Weierstrass type of statement that answers the question mentioned in the title.

The difficulties which arise are due to the following two facts:

1. The set of piecewise linear functions does not have any multiplicative structure.

2. Instead of one subspace $X \subset C[0, 1]$, we usually deal with a sequence of subspaces $X_n \subset C[0, 1]$.

Let K be a compact metric space, and let $C(K)$ denote the Banach space of all continuous real-valued functions on K. We identify the dual of $C(K)$ with the Banach space of all regular Borel measures on K:

$$C(K)^* = \mathfrak{M}(K).$$

DEFINITION 1. Let X be a Banach space. A sequence of subspaces $X_n \subset X$ is said to be *asymptotically dense* in X if, for every $x \in X$, there exists an $x_n \in X_n$ for each n such that

$$x_n \xrightarrow{s} x.$$

DEFINITION 2. A sequence of subspaces $x_n \subset C(K)$ is called a *separating sequence* if there exists a number M such that for each pair of closed disjoint subsets $F_1, F_2 \subset K$, there is a number $N := N(F_1, F_2)$ with the property that, for every $n > N$, there is an $x_n \in X_n$ with

$$x_n|_{F_2} = 0, \qquad x_n|_{F_2} = 1, \qquad \|x_n\| \leqslant M. \tag{1}$$

THEOREM 1. *Let X be a separable Banach space, and let X_n be a*

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sequence of subspaces of X. The sequence X_n is asymptotically dense in X iff, for every infinite subset $\mathbb{N}' \subset \mathbb{N}$, the conditions

$$x_n^* \in X_n^{\perp}, \qquad \|x_n^*\| \leqslant 1 \qquad (n \in \mathbb{N}') \tag{2}$$

imply

$$x_n^* \xrightarrow{w^*} 0 \qquad (n \in \mathbb{N}'). \tag{3}$$

Proof. Let (2) imply (3), and let $x \in X$ be such that $\operatorname{dist}(x, X_n) \geqslant \varepsilon > 0$ for all n from some infinite subset $\mathbb{N}' \subset \mathbb{N}$. Then, by the Hahn–Banach Theorem, there are $x_n^* \in X_n^{\perp}$ such that

$$\|x_n^*\| \leqslant \frac{1}{\varepsilon}, \qquad x_n^*(x) > 1,$$

which contradicts (3).

Conversely, let X_n be asymptotically dense in X, and let $x_n^* \in X_n^{\perp}$ satisfy (2). Without loss of generality, we may assume that $\mathbb{N}' = \mathbb{N}$. Then, there exists $\mathbb{N}_1 \subset \mathbb{N}$ such that

$$x_n^* \xrightarrow{w^*} x^* \qquad (n \in \mathbb{N}_1)$$

for some $x^* \in X^*$. We want to show that $x^* = 0$. Indeed, given $x \in X$, we can find $x_n \in X_n$ such that $x_n \to^s x$. Since $x_n^*(x_n) = 0$, we obtain

$$|x^*(x)| = \lim |x_n^*(x)| = \lim |x_n^*(x) - x_n^*(x_n)| \leqslant \lim \|x_n - x\| = 0.$$

So, zero is a cluster point for x_n^*. The same consideration shows that it is the onlyc luster point for x_n^*, and thus $x_n^* \to^{w^*} 0$. ∎

Remark. The separability of X was used only in the second part of the proof. Thus, the conditions of Theorem 1 are sufficient for any Banach space.

Next, we need the following

THEOREM 2 (cf. [1]). *Let $\mu_n \in \mathfrak{M}(K)$ be a w^*-convergent sequence. Let C_k be an increasing sequence of closed sets such that $\bigcup C_k = K$. Then,*

$$|\mu_n| \, (K \backslash C_k) \to 0 \qquad \text{as} \quad K \to \infty,$$

uniformly in n.

THEOREM 3. *Let $X_n \subset C(K)$ be a separating sequence. Then, X_n is asymptotically dense in $C(K)$.*

Proof. We have to show that the conditions

$$\mu_n \in X_n^{\perp}, \qquad \|\mu_n\| \leqslant 1 \tag{4}$$

imply $\mu_n \to^{w^*} 0$. Since the μ_n are uniformly bounded, there exists an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ and a $\mu \in \mathfrak{M}(K)$ such that $\mu_n \to^{w^*} \mu$ $(n \in \mathbb{N}_1)$. We want to show that, for every compact $C \subset K$,

$$\mu(C) = 0.$$

Let the closed sets $F_k \subset C^c := K \backslash C$ satisfy $\bigcup F_k = C^c$. Then for each k, there is a number N such that for all $n > N$, we can find an $x_n \in X$ with

$$\|x_n\| \leqslant M, \qquad x_n|_{F_k} = 0 \qquad x_n|_C = 1.$$

Since $\mu_n \in X_n^{\perp}$, we obtain

$$0 = \int_K x_n \, d\mu_n = \mu_n(C) + \int_{C^c \backslash F_k} x_n \, d\mu_n.$$

Therefore,

$$|\mu_n(C)| \leqslant M \, |\mu_n| \, (C^c \backslash F_k).$$

On the other hand, let $C_k = C \cup F_k$. Then $C^c \backslash F_k = K \backslash C_k$ and by Theorem 2, $\mu_n(C) \to 0$.

Hence, zero is a cluster point for μ_n. The same considerations show that it is the only cluster point for μ_n, and consequently

$$\mu_n \xrightarrow{w} 0. \quad \blacksquare$$

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Reference

1. R. M. Dudley, Convergence of Baire measures, *Studia Math.* **27** (1966), 251–268.